MATH2060B Midterm II Solution

1(a) $\left|\dfrac{x\cos(x)}{\sqrt{1+x^5}}\right| \le \dfrac{x}{\sqrt{x^5}} = x^{-3/2}, \forall x \in [1,\infty)$.
Since $\int_1^\infty x^{-3/2}dx$ converges, by comparison test, $\int_1^\infty \left|\dfrac{x\cos(x)}{\sqrt{1+x^5}}\right| dx$ converges.
Hence $\int_1^\infty \dfrac{x\cos(x)}{\sqrt{1+x^5}}dx$ converges.

1(b) Let $F(y) = \int_0^y f(t)dt, \forall y \in \mathbb{R}$. Then $g(x) = F(2x+1) - F(2x), \forall x \in \mathbb{R}$.
Since f is continuous, by the fundamental theorem of calculus F is differentiable and $F'(y) = f(y), \forall y \in \mathbb{R}$.
Then by chain rule g is differentiable and $g'(x) = 2f(2x+1) - 2f(2x), \forall x \in \mathbb{R}$.
By triangle inequality, $|g'(x)| \le 2(|f'(2x+1)| + |f'(2x)|) \le 2(5+5) = 20, \forall x \in \mathbb{R}$.

2(a) When $x = 0$, $f_n(0) = 0\ \forall n$, so $\lim f_n(0) = 0$.
When $0 < x < 2$, $f_n(x) = \dfrac{x^n}{2^n + x^n} = \dfrac{1}{(\frac{2}{x})^n + 1} \to 0$ as $n \to \infty$.
When $x = 2$, $f_n(2) = \frac{1}{2}\ \forall n$, so $\lim f_n(2) = \frac{1}{2}$.
Hence $\{f_n\}$ converges pointwise to a function f ,where $f(x) = \begin{cases} 0 & \text{if } x \in [0,2) \\ \frac{1}{2} & \text{if } x = 2 \end{cases}$

2(b) $||f_n - f||_{[0,1]} = \sup\left\{\left|\dfrac{x^n}{2^n + x^n} - 0\right| : x \in [0,1]\right\} \le \dfrac{1}{2^n} \to 0$ as $n \to \infty$.
Hence $\{f_n\}$ converges to f uniformly on $[0,1]$.

2(c) Suppose $\{f_n\}$ converges to f uniformly on $[0,2]$. Since each f_n is continuous on $[0,2]$, then its limit f must also be continuous on $[0,2]$.
Contradiction. Hence $\{f_n\}$ does not converge to f uniformly on $[0,2]$.

3 By uniform convergence, $\exists N \in \mathbb{N}$ s.t. $\forall n \ge N, ||f_n - f||_{[0,1]} < 1$.
$\Rightarrow ||f_n||_{[0,1]} \le ||f||_{[0,1]} + 1, \forall n \ge N$.
Let $M = \max\left\{||f_1||_{[0,1]}, ..., ||f_{N-1}||_{[0,1]}, ||f||_{[0,1]} + 1\right\}$.
Then $||f_n||_{[0,1]} \le M, \forall n \in \mathbb{N}$ and the result follows.

4(a) Let $0 = x_0 < x_1 < ... < x_n = 1$ be the partition points of P.

$$\sup_{[x_{i-1},x_i]} g - \inf_{[x_{i-1},x_i]} g = \sup_{x,y\in[x_{i-1},x_i]} |f(x)^2 - f(y)^2| = \sup_{x,y\in[x_{i-1},x_i]} |f(x)+f(y)||f(x)-f(y)|$$
$$\le 6\left(\sup_{x,y\in[x_{i-1},x_i]} |f(x)-f(y)|\right) = 6\left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f\right).$$

Then

$$U(g,P) - L(g,P) = \sum_{i=1}^n \left(\sup_{[x_{i-1},x_i]} g - \inf_{[x_{i-1},x_i]} g\right)(x_i - x_{i-1})$$
$$\le 6\sum_{i=1}^n \left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f\right)(x_i - x_{i-1}) = 6(U(f,P) - L(f,P)).$$

4(b) Given $\epsilon > 0, \exists$ a partition P of $[0,1]$ s.t. $U(f,P) - L(f,P) < \dfrac{\epsilon}{6}$.
For this P, we have $U(g,P) - L(g,P) < 6(U(f,P) - L(f,P)) < \epsilon$.
Hence g is Riemann integrable.

5 For any $N \in \mathbb{N}$, let $n > m > N$. Then
$\|g_n - g_m\|_{[0,1]}^2 = \int_0^1 \left\|(g_n - g_m)^2\right\|_{[0,1]} dx \geq \int_0^1 (g_n(x) - g_m(x))^2 dx$
$= \int_0^1 (g_n(x)^2 - 2g_n(x)g_m(x) + g_m(x)^2) dx = 2.$
By Cauchy criterion, $\{g_n\}$ does not converge uniformly on $[0,1]$.